Algebraic Number Theory

Exercise Sheet 6

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Exercise 1. Let K be a number field of degree n over $\mathbb{Q}$. Let I be a fractional ideal in $\mathcal{F}(\mathcal{O}_K)$. Show that $I \simeq \mathbb{Z}^n$ as abelian groups. Deduce that $\mathcal{O}_K/I$ is a finite group for every ideal I in $\mathcal{O}_K$.
Hint: Show that $I \cap \mathbb{Z} \neq \emptyset$.

Exercise 2. Let A be a Dedekind ring. Using localisation prove the following: if I and J are two ideals in A, such that $I + J = A$, then $IJ = I \cap J$.

Exercise 3. Let A be a Dedekind ring, S a multiplicative subset in A and $I \in \mathcal{F}(A)$. Let $I = \mathcal{P}_1^{n_1}...\mathcal{P}_k^{n_k}$ be the decomposition of I into a product of powers of prime ideals. Find the decomposition of $S^{-1}I$ in $\mathcal{F}(S^{-1}A)$.
Hint: Use the description of prime ideals in $S^{-1}A$ (Proposition L2).

Exercise 4. Let A be a Dedekind ring and $f \in A$, $f \neq 0$. Denote by S_f the multiplicative subset $\{f^n \mid n \in \mathbb{Z}, n \geq 0\}$ and by A_f the localization $S_f^{-1}A$.
(1) Show that the natural group homomorphism $C(A) \to C(A_f)$ is surjective.
(2) Show that the set $\{\mathcal{P} \in \mathrm{Spec}\,(A) | \, \mathcal{P} \cap S_f \neq \emptyset\}$ is finite. Denote the cardinality of this set by r.
(3) Show that A_f^*/A^* is a free $\mathbb{Z}$-module of rank $\leq r$.
Hint: Use the exact sequence $0 \to A^* \to K^* \to \mathcal{F}(A)$ and the same sequence for A_f.
(4) Let $n > 0$ be an integer. Deduce that $\mathbb{Z}[\frac{1}{n}]^* \simeq \{\pm 1\} \times \mathbb{Z}^r$, where r is the number of prime divisors of n. *Hint:* Note that $\mathbb{Z}[\frac{1}{n}] = A_f$ for $A = \mathbb{Z}$ and $f = n$.